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PRICING OPTIONS ON VOLATILITY OF VOLATILITY MODEL BASED ON HESTON'S STOCHASTIC VOLATILITY WITH CREDIT RISK

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Abstract. This paper is a research for pricing OTC(over-the-counter) options to entail default risk occurred by an option writer. We drive the closed-form pricing formula to approximate the prices of the so-called vulnerable options introduced by Klein [12] under the Heston stochastic volatility of volatility(vol-of-vol) model proposed in [5]. In the case of OTC options, the price of both the option writer's asset as well as the underling asset must be considered for pricing the options, unlike in the case of exchange-traded options. In this paper, we derive the partial differential equations(PDEs) whose solution is the price of the vulnerable options and solve the PDEs by asymptotic analysis [4] and the two dimensional Fourier transform method.

1. Introduction

Since appearance of the Black-Scholes model introduced by Black and Scholes [1], the researches on option pricing and implied volatility have been done actively. To overcome phenomenon of volatility smile/skew observed in financial market data, a variety of models for the price of underlying asset with stochastic volatility are studied. The local volatility model with a constant elasticity of variance(so-called, CEV model) was suggested by Cox [2]. The multiscale stochastic volatility model whose volatility is driven by a external mean-reverting stochastic process was introduced by Fouque et al. [4]. Kim et al. [10] proposed the stochastic volatility model with a stochastic elasticity of variance(so-called, SEV model) where the constant elasticity of variance is extended into an Ornstein-Uhlenbeck(OU) stochastic process. As the Heston's

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stochastic volatility model proposed by Heston [7] is a representative one in both practical and theoretical aspects among a variety of stochastic volatility models, the volatility of the model is driven by the Cox-Ingersoll-Ross(CIR) model [3].

On the other hand, since VIX(volatility index) futures and options were launched on CBOE(Chicago Board Option Exchange) in 2004 and 2006, the size of VIX derivative market increases significantly.(cf. Luo et al. [16].) The role of VVIX(volatility of volatiolity index) that means the volatility of VIX becomes important, because the VVIX is an effective tool for measuring uncertainty of volatility risk. In line with this situation, there are research results on underlying asset models with the vol-of-vol(volatility of volatility). For instant, the literatures [5, 9, 11, 15] are representative ones about the vol-of-vol model underlying the derivatives on a stock or a volatility index. Fouque et al. [5] researched the Heston's stochastic volatility model with vol-of-vol underlying index options and volatility index options, and the implied volatility under the model calculated from the S&P 500 and VIX data. Huang et al. [9] investigated the implication of the volatility of volatility on index options and volatility index options. Kim and Kim [11] dealt with variance swaps under the revised Heston's model with both stochastic elasticity volatility and vol-of-vol, and derived the pricing formula of variance swaps under the model. The Heston's stochastic volatility model with a fast mean-reverting volatility of volatility underlying the value of the variance swap was studied in the work of Lee et al. [15].

In the option contracts in exchange, an option holder is not exposed to credit risk by the counter party, while the trade of over-the-counter(OTC) market is vulnerable to the default caused by the option writer. So, in OTC market, the price of contingent claim should be considered for the prices of both the underlying asset and the counter party. Klein [12] introduced the payoff structure that an option writer owns the other liabilities and claims under the Black-Scholes model [1]. Yang et al. [19] derived the option pricing formula of the options with the payoff structure introduced by Klein [12] under the stochastic volatility model with fast mean-reversion [6]. The Heston's stochastic volatility model [7] was used for pricing the so-called vulnerable option [12] in the work of Lee et al. [13]. In the research of Lee and Kim [14], the Heston's stochastic volatility was extended to generalized multiscale Heston's stochastic volatility with a fast mean-reverting factor.

This paper is organized as follows. In section 2, we formulate the underling asset pricing model using the vol-of-vol proposed by Fouque et al. [5] and the payoff structure of the option vulnerable to credit risk introduced by Klein [12]. Section 3 provides the partial differential equations(PDEs) whose solution is the price of the option defined in section 2. In section 4, we derive the option pricing formula by the asymptotic analysis [4] and the well-known Fourier transform method. Section 5 concludes this paper.

2. Model framework and option payoff structure

In OTC market, the contract execution by the option writer can not be guaranteed at the expiration date T. i.e, the contingent claim is exposed to credit risk. So, contrary to exchange trading, in OTC market trading, it needs that the payoff structure of option reflects the price of option writer's asset as well as the price of its underlying asset. In this paper, we introduce the option framework suggested by Klein [12] with the payoff structure depending on the prices of both the underlying asset and the counterparty's asset.

We assume that the underlying asset S_t and the option writer's asset V_t follow the dynamics of the stochastic processes suggested by Fouque and Saporito [5]. Under a risk-neutral measure $\mathbb{Q}^*$, the stochastic process S_t is the price of the underlying asset whose volatility is driven by the generalized CIR process U_t. Moreover, the volatility U_t is the stochastic process of which volatility is driven by the fast and slow mean-reversion stochastic processes Y_t and Z_t, respectively. The price V_t of the option write's asset is also in the form of a generalized Heston's stochastic volatility model of which volatility of volatility is driven by the Ornstein-Uhlenbeck processes Y_t and Z_t. i.e., the prices of the underlying asset and the option writer's asset are govern by the stochastic differential equations(SDEs) given by

$$
\begin{aligned}
\frac{dS_t}{S_t} &= (r-q)\,dt + \sigma_s\sqrt{U_t}\,dW_t^s, \\
dU_t &= \kappa\,(m-U_t)\;dt + f(Y_t,Z_t)\sqrt{U_t}\,dW_t^u, \\
dY_t &= \frac{U_t}{\epsilon}\,a(Y_t)\,dt + \sqrt{\frac{U_t}{\epsilon}}\,b\,(Y_t)\;dW_t^y, \\
dZ_t &= \delta U_t\,c(Z_t)\,dt + \sqrt{\delta U_t}\,d(Z_t)\,dW_t^z
\end{aligned}
\tag{2.1}
$$

and

$$
\frac{dV_t}{V_t} = (r-q)\,dt + \sigma_v\sqrt{U_t}\,dW_t^v,
$$

where the small constants ϵ and δ satisfy the condition such that the condition $0 < \delta < \epsilon < \sqrt{\delta} \ll 1$. The values of the parameters ϵ and δ determine the mean-reverting rate of the stochastic processes Y_t and Z_t, respectively. Here, the standard Brownian motions W_t^s, W_t^u, W_t^y, W_t^z and W_t^v are correlated by

$$\begin{aligned}
&d\langle W_s, W_u\rangle = \rho_{su}\, dt, \quad d\langle W_s, W_y\rangle = \rho_{sy}\, dt,\\
&d\langle W_s, W_z\rangle = \rho_{sz}\, dt, \quad d\langle W_s, W_v\rangle = \rho_{sv}\, dt,\\
&d\langle W_u, W_y\rangle = \rho_{uy}\, dt, \quad d\langle W_u, W_z\rangle = \rho_{uz}\, dt,\\
&d\langle W_u, W_v\rangle = \rho_{uv}\, dt, \quad d\langle W_y, W_z\rangle = \rho_{yz}\, dt,\\
&d\langle W_y, W_v\rangle = \rho_{yv}\, dt, \quad d\langle W_z, W_v\rangle = \rho_{zv}\, dt.
\end{aligned}$$

It is assumed that the mean-reverting stochastic process Y_t has an unique invariant distribution denoted by $\Phi(\cdot)$ and the functions $a(\cdot)$, and $b(\cdot)$ are smooth. $f(\cdot,\cdot)$ is assumed to be a smooth and bounded function such that the average $\left\langle f^2(\cdot, z)\right\rangle$ is bounded for any real number z. Here, the notation $\langle\cdot\rangle$ means the average with respect to the probability density function $\Phi(\cdot)$ defined by

$$\langle g(\cdot)\rangle := \int_{-\infty}^{\infty} g(y)\, \Phi(y) dy.$$

A feature that appeared in an OTC market is that contracts of contingent claim have possibility of default. For instant, in case of European option traded in exchange, if an option holder exercise the contingent claim at the maturity T, the contract execution can be guaranteed by exchange. However, even though, in an OTC market, the option holder wants to exercise the contingent claim at the maturity T, there is a possibility that the contract may go bankrupt by the option writer. So, it's necessary that the pricing of the contingent claim traded in an OTC market reflects the price V_T of option writer's asset at the maturity T. Klein [12] proposed the options associasted with credit risk where the payoff function $h(\cdot,\cdot)$ depending on the asset prices (S_T, V_T) at the maturity T is defined by

$$h\left(s, v\right) = \max(s-K, 0)\left(1_{\{v\geq D^*\}}(v) + 1_{\{v<D^*\}}(v)\frac{(1-\alpha)v}{D}\right). \tag{2.2}$$

Here, 1_A is the indicator function such that $1_A(x)$ becomes 1 if and only if x is included in a set A, and D^* is the decision level to determine whether the contract is default or not. The amount D, which represents liabilities owned by the option writer, and the constant α, which represents the weight ratio of loss incurred due to default or bankruptcy, determines that the option holder receives the payment proportional to $\frac{(1-\alpha)}{D}V_T$.

3. Derivative pricing and PDE problems

In this section, we define the option price with the payoff (2.2) and derive the partial differential equations(PDEs) whose solution gives the option price.

Under the risk-neutral measure $\mathbb{Q}^*$, the price P of the vulnerable option is defined by

$$P(t,s,v,u,y,z) = \mathbb{E}^* \left[e^{-r\,(T-t)}\, h\,(s,v) \,\Big|\, S_t = s,\, V_t = v,\, U_t = u,\, Y_t = y,\, Z_t = z \right], \tag{3.1}$$

where K is a strike price, r is a risk-free interest rate and h is the payoff function defined in (2.2). It has known that the price P (3.1) in the form of the expectation can be a PDE problem by the Feynman-Kac formula(see Oksendel [17]). Applying the Feynman-Kac formula into the price (3.1), we can gain the PDE problem given by

$$\begin{aligned} &\left(\begin{array}{l} \frac{1}{\epsilon}\, u\,\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\, u\,\mathcal{L}_1 + \mathcal{L}_2 \\ + \frac{\sqrt{\delta}}{\sqrt{\epsilon}}\, u\,\mathcal{M}_0 + \sqrt{\delta}\, u\,\mathcal{M}_1 + \delta\, u\,\mathcal{M}_2 \end{array} \right) P(t,x,v,u,y,z) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad P(T,s,v,u,y,z) = h(s,v), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \mathcal{L}_0 &:= a(y)\,\frac{\partial}{\partial y} + \frac{1}{2}\, b^2(y)\,\frac{\partial^2}{\partial y^2}, \\ \mathcal{L}_1 &:= \rho_{sy}\,\sigma_s\, b(y)\, s\,\frac{\partial^2}{\partial s \partial y} + \rho_{uy}\, f(y,z)\, b(y)\,\frac{\partial^2}{\partial u \partial y} + \rho_{vy}\,\sigma_v\, b(y)\, v\,\frac{\partial^2}{\partial v \partial y}, \\ \mathcal{L}_2 &:= \frac{\partial}{\partial t} + (r-q)\left(s\,\frac{\partial}{\partial s} + v\,\frac{\partial}{\partial v} \right) + \kappa\,(m-u)\,\frac{\partial}{\partial u} \\ &\quad + \frac{1}{2}\, u\left(\sigma_s^2\, s^2\,\frac{\partial^2}{\partial s^2} + f^2(y,z)\,\frac{\partial^2}{\partial u^2} + \sigma_v^2\, v^2\,\frac{\partial^2}{\partial v^2} \right) - r \\ &\quad + u\left(\rho_{su}\,\sigma_s\, f(y,z)\, s\,\frac{\partial^2}{\partial s \partial u} + \rho_{sv}\,\sigma_s\,\sigma_v\, s\, v\,\frac{\partial^2}{\partial s \partial v} + \rho_{vu}\, f(y,z)\,\sigma_v\, v\,\frac{\partial^2}{\partial v \partial u} \right), \\ \mathcal{M}_0 &:= \rho_{yz}\, b(y)\, d(z)\,\frac{\partial^2}{\partial y \partial z}, \end{aligned}$$

$$\mathcal{M}_1 := \rho_{sz}\,\sigma_s\,d(z)\,s\,\frac{\partial^2}{\partial s\partial z} + \rho_{uz}\,f(y,z)\,d(z)\,\frac{\partial^2}{\partial u\partial z} + \rho_{vz}\,\sigma_v\,d(z)\,v\,\frac{\partial^2}{\partial v\partial z},$$

$$\mathcal{M}_2 := c(z)\,\frac{\partial}{\partial z} + \frac{1}{2}\,d^2(z)\,\frac{\partial^2}{\partial z^2}$$

Now and for, we derive the closed-form pricing formula to approximate the solution of the PDE (3.2).

4. Asymptotic expansion and Fourier transform

In this section, we introduce the series expansion (4.1) to solve the PDE (3.2), and solve the PDEs by the two dimensional Fourier transform method and the asymptotic method introduced by Fouque et al. [4]. Then we can approximate the solution $P(t,s,v,u,y,z)$ using the zero and first order terms in the series expansion given by

$$P(t,s,v,u,y,z) = \sum_{i,j=0}^{\infty} \epsilon^{i/2}\delta^{j/2} P_{ij}(t,s,v,u,y,z). \tag{4.1}$$

It has known that the solution $\tilde{P}^*$ approximated by the zero and first order terms has the accuracy given by

$$\Big| P(t,s,v,u,y,z) - \tilde{P}^*(t,s,v,u,y,z) \Big| < C\,(\epsilon + \delta)$$

for some constant C, where the function $\tilde{P}^*$ is defined by

$$\begin{aligned}\tilde{P}^*(t,s,v,u,y,z) :=& P_{00}(t,s,v,u,y,z) + \sqrt{\epsilon}P_{10}(t,s,v,u,y,z)\\ &+ \delta\epsilon P_{01}(t,s,v,u,y,z).\end{aligned}$$

The above accuracy of the solution $\tilde{P}^*$ to the price $P(t,s,v,u,y,z)$ has already known in the work of Fouque et al. [4, 5].

Now, putting the series expansion (4.1) into the PDE (3.2), we can obtain the equation given by

$$\begin{aligned}&\frac{1}{\epsilon}\,u\,\Big[\mathcal{L}_0 P_{00} + \sqrt{\delta}\mathcal{L}_0 P_{01} + \delta\mathcal{L}_0 P_{02}\Big]\\ &\quad + \frac{1}{\sqrt{\epsilon}}\,u\,\Big[\mathcal{L}_0 P_{10} + \mathcal{L}_1 P_{00} + \sqrt{\delta}\,(\mathcal{L}_0 P_{11} + \mathcal{L}_1 P_{01} + \mathcal{M}_0 P_{00})\\ &\qquad\qquad + \delta\,(\mathcal{L}_0 P_{12} + \mathcal{L}_1 P_{02} + \mathcal{M}_0 P_{01})\Big]\end{aligned}$$

(4.2)

$$
\begin{aligned}
&+ u \Bigg[\mathcal{L}_0 P_{20} + \mathcal{L}_1 P_{10} + \frac{1}{u}\,\mathcal{L}_2 P_{00} \\
&\qquad + \sqrt{\delta}\Big(\mathcal{L}_0 P_{21} + \mathcal{L}_1 P_{11} + \frac{1}{u}\,\mathcal{L}_2 P_{01} + \mathcal{M}_0 P_{10} + \mathcal{M}_1 P_{00} \Big) \\
&\qquad\quad + \delta \Big(\mathcal{L}_0 P_{22} + \mathcal{L}_1 P_{12} + \frac{1}{u}\,\mathcal{L}_2 P_{02} + \mathcal{M}_0 P_{11} + \mathcal{M}_1 P_{01} + \mathcal{M}_2 P_{00} \Big) \Bigg] \\
&+ \sqrt{\epsilon}\, u \Bigg[\mathcal{L}_0 P_{30} + \mathcal{L}_1 P_{20} + \frac{1}{u}\,\mathcal{L}_2 P_{10} \\
&\qquad + \sqrt{\delta}\Big(\mathcal{L}_0 P_{31} + \mathcal{L}_1 P_{21} + \frac{1}{u}\,\mathcal{L}_2 P_{11} + \mathcal{M}_0 P_{20} + \mathcal{M}_1 P_{10} \Big) \\
&\qquad\quad + \delta \Big(\mathcal{L}_0 P_{32} + \mathcal{L}_1 P_{22} + \frac{1}{u}\,\mathcal{L}_2 P_{12} + \mathcal{M}_0 P_{21} + \mathcal{M}_1 P_{11} + \mathcal{M}_2 P_{10} \Big) \Bigg] \\
&+ \epsilon\, u \Bigg[\mathcal{L}_0 P_{40} + \mathcal{L}_1 P_{30} + \frac{1}{u}\,\mathcal{L}_2 P_{20} \\
&\qquad + \sqrt{\delta}\Big(\mathcal{L}_0 P_{41} + \mathcal{L}_1 P_{31} + \frac{1}{u}\,\mathcal{L}_2 P_{21} + \mathcal{M}_0 P_{30} + \mathcal{M}_1 P_{20} \Big) \\
&\qquad\quad + \delta \Big(\mathcal{L}_0 P_{42} + \mathcal{L}_1 P_{32} + \frac{1}{u}\,\mathcal{L}_2 P_{22} + \mathcal{M}_0 P_{31} + \mathcal{M}_1 P_{21} + \mathcal{M}_2 P_{20} \Big) \Bigg] \\
&+ \cdots \\
&= 0
\end{aligned}
$$

with the boundary condition

$$
\sum_{i,j=0}^{\infty} \epsilon^{i/2} \delta^{j/2} P_{ij}(T, s, v, u, y, z) = h(s, v). \tag{4.3}
$$

The following lemma provides a robust theoretic foundation for deriving the PDEs for each function P_{ij} in the series expansion (4.1).

LEMMA 4.1. *Consider the equation* $\mathcal{L}_0\, \mathcal{Q}(t, s, v, u, y, z) = \mathcal{P}(t, s, v, u, y, z)$ *with the infinitesimal generator* $\mathcal{L}_0$ *of the stochastic process* Y_t *in (2.1). Then it is the existent necessary and sufficient condition for the solution* $\mathcal{Q}$ *that the function* $\mathcal{P}$ *satisfies* $\langle \mathcal{P}(t, s, v, u, \cdot, z)\rangle = 0$. *Moreover, assume that* $\mathcal{P} = 0$ *and the derivative of* $\mathcal{P}$ *with respect to the variable* y *does not grow exponentially, nevertheless the variable* y *goes to infinity. Then the function* $\mathcal{Q}$ *is independent on the variable* y.

Proof. Refer to Fouque et al. [4] and Ramm [18]. □

To derive the first order term P_{10}, it is essential to first determine the second order term P_{20}. In the following lemma, we derive P_{20} from (4.2) by using Lemma 4.1.

LEMMA 4.2. *The second order term P_{20} depends on the variable y and is the solution in the form given by*

$$P_{20}(t,s,v,u,y,z) = -\begin{pmatrix} \frac{1}{2}\phi(y,z)\frac{\partial^2}{\partial u^2} \\ +\rho_{su}\,\sigma_s\,\psi(y,z)\,s\,\frac{\partial^2}{\partial s \partial u} \\ +\rho_{vu}\,\sigma_v\,\psi(y,z)\,v\,\frac{\partial^2}{\partial v \partial u} \end{pmatrix} P_{00}(t,s,v,u,y,z) + C(t,s,v,u,z), \tag{4.4}$$

where $C(t,s,v,u,z)$ is an unspecified function that is constant with respect to the variable y, and the functions $\phi(y,z)$ and $\psi(y,z)$ are the solutions of the equations defined by

$$\mathcal{L}_0\phi(y,z) = f^2(y,z) - \left\langle f^2(\cdot,z)\right\rangle \quad \text{and} \quad \mathcal{L}_0\psi(y,z) = f(y,z) - \left\langle f(\cdot,z)\right\rangle. \tag{4.5}$$

Proof. Applying Lemma 4.1 into the $\frac{\delta^{\,i/2}}{\epsilon}$ $(i=0,1,2)$ terms, $P_{0,i}$ $(i=0,1,2)$ is independent on the variable y and so we have the equalities $\mathcal{L}_1 P_{10} = 0$ and $\langle\mathcal{L}_2\rangle\, P_{00} = 0$. Lemma 4.1 yields that $\mathcal{O}(1)$ term in (4.2) becomes the Poisson equation

$$\begin{aligned} \mathcal{L}_0 P_{20} &= -\frac{1}{u}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\right) P_{00} \\ &= -\begin{bmatrix} \frac{1}{2}\left(f^2(y,z) - \left\langle f^2(\cdot,z)\right\rangle\right)\frac{\partial^2}{\partial u^2} \\ +\rho_{su}\sigma_s\left(f(y,z) - \langle f(\cdot,z)\rangle\right) s\,\frac{\partial^2}{\partial s \partial u} \\ +\rho_{vu}\sigma_v\left(f(y,z) - \langle f(\cdot,z)\rangle\right) v\,\frac{\partial^2}{\partial v \partial u} \end{bmatrix} P_{00}, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned}\mathcal{H} &:= \langle \mathcal{L}_2 \rangle \\ &= \frac{\partial}{\partial t} + (r-q)\left(s\,\frac{\partial}{\partial s} + v\,\frac{\partial}{\partial v}\right) + \kappa\,(m-u)\,\frac{\partial}{\partial u} \\ &\quad + \frac{1}{2}\,u\left(\sigma_s^2\,s^2\,\frac{\partial^2}{\partial s^2} + \bar{\sigma}^2(z)\,\frac{\partial^2}{\partial u^2} + \sigma_v^2\,v^2\,\frac{\partial^2}{\partial v^2}\right) - r \\ &\quad + u\left(\rho_{su}\,\sigma_s\,\tilde{\sigma}(z)\,s\,\frac{\partial^2}{\partial s \partial u} + \rho_{sv}\,\sigma_s\,\sigma_v\,s\,v\,\frac{\partial^2}{\partial s \partial v} + \rho_{vu}\,\tilde{\sigma}(z)\,\sigma_v\,v\,\frac{\partial^2}{\partial v \partial u}\right),\end{aligned}$$

$\bar{\sigma}(z) := \left(\left\langle f^2(\cdot,z)\right\rangle\right)^{1/2}$, and $\tilde{\sigma}(z) := \langle f(\cdot,z)\rangle$.

From (4.5) and (4.6), we can have the solution (4.4) for a constant $C(t,s,v,u,z)$ with respect to the variable y. □

From the following proposition, we derive the PDEs whose solutions are the zero and first order terms P_{ij}, $(i,j) \in \{(0,0),(1,0),(0,1)\}$ in (4.1).

PROPOSITION 4.3. *The zero and first order terms P_{ij}, $(i,j) \in \{(0,0)$ $(1,0),(0,1)\}$ are independent on the variable y, i.e., $P_{ij} = P_{ij}(t,s,v,u,z)$ and the solutions of the PDEs given, for $(i,j) \in \{(0,0),(1,0),(0,1)\}$, by*

$$\begin{aligned}\mathcal{H}P_{ij}(t,s,v,u,z) &= G_{ij}(t,s,v,u,z) \\ P_{ij}(T,s,v,u,z) &= H_{ij}(s,v)\end{aligned} \tag{4.7}$$

where $G_{ij}(t,s,v,u,z)$ and $H_{ij}(s,v)$ are defined by

$$G_{ij}(t,s,v,u,z) = \begin{cases} 0 & ,\ (i,j)=(0,0) \\ -u\,\langle \mathcal{L}_1 P_{20}(t,s,v,u,\cdot,z)\rangle & ,\ (i,j)=(1,0) \\ -u\,\langle \mathcal{M}_1\rangle\,P_{00}(t,s,v,u,z) & ,\ (i,j)=(0,1) \end{cases},$$

$$\text{and}\quad H_{ij}(s,v) = \begin{cases} h(s,v) & ,\ (i,j)=(0,0) \\ 0 & ,\ (i,j)\in\{(1,0),(0,1)\} \end{cases}.$$

Proof. In the proof of Lemma 4.2, we found that P_{00} and P_{01} are independent on the variable y, and P_{00} is the solution of the PDE $\langle \mathcal{L}_2\rangle\,P_{00} = 0$. Lemma 4.1 yields that the $\frac{1}{\sqrt{\epsilon}}$ term in (4.2) becomes $\mathcal{L}_0 P_{10} = 0$, and then P_{10} is also independent on the variable y. Thus, we can have the PDE $\mathcal{H}P_{10} = -u\,\langle \mathcal{L}_1 P_{20}\rangle$ from the $\sqrt{\epsilon}$ term in (4.2).

On the other hand, since $\mathcal{L}_1$ and $\mathcal{M}_0$ are the derivatives with respect to the variable y, the $\frac{\sqrt{\delta}}{\sqrt{\epsilon}}$ term in (4.2) becomes $\mathcal{L}_0 P_{11} = 0$, and so P_{11} is independent on the variable y. Hence, we can derive the PDE $u\mathcal{L}_0 P_{21} + \mathcal{L}_2 P_{01} + u\mathcal{M}_1 P_{00} = 0$ to be $\mathcal{H}P_{01} = -u\,\langle \mathcal{M}_1 \rangle\, P_{00}$, from the $\sqrt{\delta}$ term, by Lemma 4.1. Moreover, the boundary condition $H_{ij}(s,v)$ at the maturity T can be obtained by applying (4.1) into (4.3). □

In the following proposition, we solve the PDEs (4.7) of P_{ij}, $(i,j) \in \{(0,0),(1,0),(0,1)\}$ by the Fourier transform method(cf. Lee et al. [13], and Lee and Kim [14]).

PROPOSITION 4.4. *The zero order term P_{00} and the first order terms P_{ij}, $(i,j) \in \{(1,0),(0,1)\}$ are provided by*

$$P_{ij}(t,s,v,u,z) = \frac{e^{-r\,\tau}}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{-i(kx+lw)}\, \tilde{A}(\tau,k,l,u,z)\, \tilde{B}_{ij}(\tau,k,l,u,z)\, \hat{h}(k,l)\, dk\, dl,$$

where the variables τ, x and v are defined by

(4.8)
$$\tau(t) = T - t, \quad x(t,s) = (r-q)\,\tau(t) + \ln s, \quad w(\tau,v) = (r-q)\,\tau(t) + \ln v,$$

the functions $\hat{h}$, $\tilde{A}$ and $\tilde{B}_{ij}$, $(i,j) \in \{(1,0),(0,1)\}$ are given by

(4.9)
$$\hat{h}(k,l) = -\frac{K^{1+ik}}{k^2 - ik}\,(D^*)^{il}\left(\frac{(1-\alpha)}{D}\frac{D^*}{il+1} - \frac{1}{il}\right), \ (k > 1, 0 < l < 1)$$
$$\tilde{A}(\tau,k,l,u,z) = e^{\,A^0(\tau,k,l,z) + u\,A^1(\tau,k,l,z)}$$
$$\tilde{B}_{ij}(\tau,k,l,u,z) = \begin{cases} 1 & ,(i,j) = (0,0) \\ \sum_{n=0}^{i+2j} u^n B^{\,n\,|\,ij}(\tau,k,l,z) & ,(i,j) = \{(1,0),(0,1)\} \end{cases},$$

and A^n, $n \in \{0,1\}$ and $B^{n\,|\,ij}$, $n \in \{0,1,\cdots,i+2j\}$, $(i,j) \in \{(1,0),(0,1)\}$ in (4.9) are appeared in Appendix.

Proof. First of all, the PDE of P_{00} in (4.7) is same with the case of the price of the vulnerable option on the Heston's stochastic volatility model [7]. The solution P_{00} has already been obtained in the work of Lee et al. [13], so we omit the detail processes for solving the PDE of P_{00} in (4.7). It remains to derive the solutions P_{ij}, $(i,j) \in \{(1,0),(0,1)\}$ satisfying the PDEs (4.7) obtained in Proposition 4.3. Changing the variables t, s, v and P_{ij} by τ, x, w and $\check{P}_{ij}(\tau,x,w,u,z) := e^{r\,\tau(t)} P_{ij}(t,s,v,u,z)$, respectively, such as the substitutions (4.8) and plugging the function

P_{20} obtained in Lemma 4.2 into the PDEs (4.7), we can have the PDEs of $\check{P}_{ij},\ (i,j),\ \in\{(1,0),(0,1)\}$ given by

(4.10)

$$\begin{array}{l}\check{\mathcal{H}}\check{P}_{ij}(\tau,x,w,u,z)=u\check{\mathcal{N}}_{ij}\check{P}_{00}(\tau,x,w,u,z)\\ \check{P}_{ij}(0,x,w,u,z)=0\end{array},\ (i,j)\in\{(1,0),(0,1)\},$$

where

$$\begin{aligned}\check{\mathcal{H}}:=&-\frac{\partial}{\partial\tau}+\kappa\,(m-u)\,\frac{\partial}{\partial u}\\&+u\left(\rho_{su}\,\sigma_s\,\tilde{\sigma}(z)\,\frac{\partial^2}{\partial x\partial u}+\rho_{sv}\,\sigma_s\,\sigma_v\,\frac{\partial^2}{\partial x\partial w}+\rho_{vu}\,\sigma_v\,\tilde{\sigma}(z)\,\frac{\partial^2}{\partial w\partial u}\right)\\&+\frac{1}{2}\,u\left[\sigma_s^2\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)+\bar{\sigma}^2(z)\,\frac{\partial^2}{\partial u^2}+\sigma_v^2\left(\frac{\partial^2}{\partial w^2}-\frac{\partial}{\partial w}\right)\right]\cdot,\end{aligned}$$

$$\check{\mathcal{N}}_{ij}=\begin{cases}\left[\begin{array}{l}\frac{1}{2}\begin{pmatrix}\rho_{sy}\sigma_s\left\langle b(\cdot)\frac{\partial\phi(\cdot,z)}{\partial y}\right\rangle\frac{\partial^3}{\partial x\partial u^2}\\+\rho_{uy}\left\langle f(\cdot,z)b(\cdot)\frac{\partial\phi(\cdot,z)}{\partial y}\right\rangle\frac{\partial^3}{\partial u^3}\\+\rho_{vy}\sigma_v\left\langle b(\cdot)\frac{\partial\phi(\cdot,z)}{\partial y}\right\rangle\frac{\partial^3}{\partial w\partial u^2}\end{pmatrix}\\+\left\langle b(\cdot)\frac{\partial\psi(\cdot,z)}{\partial y}\right\rangle\left[\begin{array}{l}\rho_{sy}\rho_{su}\sigma_s^2\frac{\partial^2}{\partial x^2}\\+\sigma_s\sigma_v\left(\rho_{vy}\rho_{su}+\rho_{sy}\rho_{vu}\right)\frac{\partial^2}{\partial x\partial w}\\+\rho_{vy}\rho_{vu}\sigma_v^2\frac{\partial^2}{\partial w^2}\end{array}\right]\frac{\partial}{\partial u}\\+\left\langle f(\cdot,z)b(\cdot)\frac{\partial\psi(\cdot,z)}{\partial y}\right\rangle\rho_{uy}\left(\rho_{su}\sigma_s\frac{\partial}{\partial x}+\rho_{vu}\sigma_v\frac{\partial}{\partial w}\right)\frac{\partial^2}{\partial u^2}\end{array}\right] & ,(i,j)=(1,0)\\ -d(z)\left(\rho_{sz}\sigma_s\frac{\partial}{\partial x}+\rho_{uz}\tilde{\sigma}(z)\frac{\partial}{\partial u}+\rho_{vz}\sigma_v\frac{\partial}{\partial w}\right)\frac{\partial}{\partial z} & ,(i,j)=(0,1)\end{cases}$$

Now, for applying the Fourier transform into (4.10), we define the functions $\hat{P}_{ij}$ by

$$\hat{P}_{ij}(\tau,k,l,u,z)=\iint_{\mathbb{R}^2}e^{i(kx+lw)}\check{P}_{ij}(\tau,x,w,u,z)\,dx\,dew \tag{4.11}$$

and then, by the inverse Fourier transform, (4.11) becomes

$$\check{P}_{ij}(\tau, x, w, u, z) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{-i(kx+lw)} \hat{P}_{ij}(\tau, k, l, u, z)\, dk\, dl. \tag{4.12}$$

Applying (4.12) into the PDEs (4.10), we can have the PDEs

$$\begin{array}{l} \hat{\mathcal{H}}\hat{P}_{ij}(\tau, k, l, u, z) = u\hat{N}_{ij}\hat{P}_{00}(\tau, k, l, u, z) \\ \hat{P}_{ij}(0, k, l, u, z) = 0 \end{array}, \ (i,j) \in \{(1,0),(0,1)\}, \tag{4.13}$$

where the operators $\hat{\mathcal{H}}$ and $\hat{\mathcal{N}}_{ij}$, $(i,j) \in \{(1,0),(0,1)\}$ are given by

$$\begin{aligned} \hat{\mathcal{H}} = & -\frac{\partial}{\partial \tau} + \kappa\,(m-u)\,\frac{\partial}{\partial u} \\ & - u\left(\rho_{su}\,\sigma_s\,\tilde{\sigma}(z)\,(ik)\,\frac{\partial}{\partial u} + \rho_{sv}\,\sigma_s\,\sigma_v\,(kl) + \rho_{vu}\,\sigma_v\,\tilde{\sigma}(z)\,(il)\,\frac{\partial}{\partial u}\right) \\ & - \frac{1}{2}\,u\left(\sigma_s^2\,\left(k^2 - ik\right) - \bar{\sigma}^2(z)\,\frac{\partial^2}{\partial u^2} + \sigma_v^2\,\left(l^2 - il\right)\right), \end{aligned}$$

$$\hat{\mathcal{N}}_{ij} = \begin{cases} -\left[\begin{array}{l} \frac{1}{2}\left(\begin{array}{l} \rho_{sy}\sigma_s \left\langle b(\cdot)\frac{\partial\phi(\cdot,z)}{\partial y}\right\rangle (ik)\,\frac{\partial^2}{\partial u^2} \\ -\rho_{uy}\left\langle f(\cdot,z)b(\cdot)\frac{\partial\phi(\cdot,z)}{\partial y}\right\rangle \frac{\partial^3}{\partial u^3} \\ +\rho_{vy}\sigma_v\left\langle b(\cdot)\frac{\partial\phi(\cdot,z)}{\partial y}\right\rangle (il)\,\frac{\partial^2}{\partial u^2} \end{array}\right) \\ + \left\langle b(\cdot)\frac{\partial\psi(\cdot,z)}{\partial y}\right\rangle \left[\begin{array}{l} \rho_{sy}\rho_{su}\sigma_s^2\left(k^2\right) \\ +\sigma_s\sigma_v\left(\rho_{vy}\rho_{su} + \rho_{sy}\rho_{vu}\right)(kl) \\ +\rho_{vy}\rho_{vu}\sigma_v^2\left(l^2\right) \end{array}\right]\frac{\partial}{\partial u} \\ + \left\langle f(\cdot,z)b(\cdot)\frac{\partial\psi(\cdot,z)}{\partial y}\right\rangle \rho_{uy}\left(\rho_{su}\sigma_s\,(ik) + \rho_{vu}\sigma_v\,(il)\right)\frac{\partial^2}{\partial u^2} \end{array}\right] & ,\ (i,j) = (1,0) \\ d(z)\left(\rho_{sz}\sigma_s(ik) - \rho_{uz}\tilde{\sigma}(z)\frac{\partial}{\partial u} + \rho_{vz}\sigma_v(il)\right)\frac{\partial}{\partial z} & ,\ (i,j) = (0,1) \end{cases}.$$

Here, consider the the solution $\hat{P}_{ij}$, $(i,j) \in \{(1,0),(0,1)\}$ has the form of

$$\hat{P}_{ij} = \sum_{n=0}^{i+2j} B^{n\,|\,ij}\,\tilde{A}(\tau, k, l, u, z)\,\hat{h}(k, l), \tag{4.14}$$

Putting (4.14) into the PDEs (4.13), we can find that the functions $B^{n\,|\,ij}$, $(i,j) \in \{(1,0),(0,1)\}$, $n \in \{0,\cdots,i+2j\}$ are the solutions of the ODEs given by

(4.15)

$$\begin{cases} \left[\frac{\partial}{\partial \tau} + n\left(\eta(k,l,z) - \bar{\sigma}^2(z)\, A^1\left(\tau,k,l,z\right)\right)\right] B^{n\,|\,ij}(\tau,k,l,z) \\ = R^{n\,|\,ij}(\tau,k,l,z) \\ B^{n\,|\,ij}(0,k,l,z) = 0 \\ \frac{\partial}{\partial \tau} B^{0\,|\,ij}(\tau,k,l,z) = \kappa m B^{1\,|\,ij}(\tau,k,l,z) \\ B^{0\,|\,ij}(0,k,l,z) = 0 \end{cases}.$$

As the ODEs (4.15) are the well-known Riccati equations, the solution method has already known well. So, we directly provide the solutions without the detail solution process, since there are many references for the Riccati equations. For instant, see Hille [8]. □

5. Conclusion

This paper is a research for the Heston's stochastic vol-of-vol model underlying the options vulnerable to credit risk. In contrast to listed options, the OTC options has the possibility of exposure to default by the option writer. So it's necessary for the OTC options to have the payoff structure determined by the underlying asset as well as the counterparty asset. In this paper, we focused on obtaining the corrected price caused by the vol-of-vol structure based on the Heston's stochastic volatility derived by the closed-form formula corrected to the Heston's stochastic volatility. A possible avenue for an extension of future work is to enhance the accuracy of approximation by deriving the high order terms.

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Appendix: the functions $A^n,\ n\in\{0,1\}$ and $B^{n\,|\,ij},\ n\in\{0,1,\cdots,i+2j\},\ (i,j)\in\{(1,0),(0,1)\}$ in (4.9) appeeared in Proposition 4.3

5.1. The functions $A^n(\tau,k,l,z),\ n=0,1,$ $\eta(k,l,z)$, $\zeta(k,l,z)$ and $\xi(k,l,z)$ appeared in the zero order term P_{00} of the series expansion (4.1)

$$
\begin{aligned}
A^1(\tau,k,l,z) &= \frac{\eta(k,l,z)+\zeta(k,lz)}{\bar{\sigma}^2(z)}\cdot\frac{1-e^{\tau\zeta(\tau,k,l,z)}}{1-\xi(k,l,z)e^{\tau\zeta(\tau,k,l,z)}},\\
A^0(\tau,k,l,z) &= \frac{\kappa m}{\bar{\sigma}^2(z)}\left[\left(\eta(\tau,k,l,z)+\zeta(\tau,k,l,z)\right)\tau-2\ln\left(\frac{1-\xi(\tau,k,l,z)e^{\tau\zeta(\tau,k,l,z)}}{1-\xi(\tau,k,l,z)}\right)\right]\\
\eta(k,l,z) &= \kappa+\rho_{su}\sigma_s\tilde{\sigma}(z)(ik)+\rho_{vu}\sigma_v\tilde{\sigma}(z)(il),\\
\zeta(k,l,z) &= \sqrt{\eta^2(k,l)+\bar{\sigma}^2(z)\left(\sigma_s^2\left(k^2-ik\right)+2\rho_{sv}\sigma_s\sigma_v(kl)+\sigma_v^2\left(l^2-il\right)\right)},\\
\xi(k,l,z) &= \frac{\eta(k,l,z)+\zeta(k,l,z)}{\eta(k,l,z)-\zeta(k,l,z)},
\end{aligned}
$$

5.2. The functions $B^{n\,|\,10}(\tau,k,l,z),\ n=0,1$ and $R^{1\,|\,10}(\tau,k,l,z)$ appeared in the first order term P_{10} of the series expansion (4.1)

$$
\begin{aligned}
B^{1\,|\,10}(\tau,k,l,z) &= \int_0^\tau e^{(\tau-s)\zeta(k,l,z)}\left(\frac{1-\xi(k,l,z)e^{s\,\zeta(k,l,z)}}{1-\xi(k,l,z)e^{\tau\,\zeta(k,l,z)}}\right)^2 R^{1\,|\,10}(s,k,l,z)\,ds,\\
B^{0\,|\,10}(\tau,k,l,z) &= \kappa m\int_0^\tau B^{1\,|\,10}(s,k,l,z)ds\\
R^{1\,|\,10}(\tau,k,l,z) &= \frac{1}{2}\begin{pmatrix}
\rho_{sy}\sigma_s\,(ik)\left\langle b\,(\cdot)\,\frac{\partial\phi(\cdot,z)}{\partial y}\right\rangle\left(A^1\,(\tau,k,l,z)\right)^2\\
-\rho_{uy}\left\langle f\,(\cdot,z)\,b\,(\cdot)\,\frac{\partial\phi(\cdot,z)}{\partial y}\right\rangle\left(A^1\,(\tau,k,l,z)\right)^3\\
+\rho_{vy}\sigma_v\,(il)\left\langle b\,(\cdot)\,\frac{\partial\phi(\cdot,z)}{\partial y}\right\rangle\left(A^1(\tau,k,l,z)\right)^2
\end{pmatrix}\\
&\quad+\left(\rho_{sy}\rho_{su}\sigma_s^2\left(k^2\right)+\rho_{vy}\rho_{vu}\sigma_v^2\left(l^2\right)\right)\left\langle b\,(\cdot)\,\frac{\partial\psi\,(\cdot,z)}{\partial y}\right\rangle A^1\,(\tau,k,l,z)\\
&\quad+\rho_{uy}\left(\rho_{su}\sigma_s\,(ik)+\rho_{vu}\sigma_v\,(il)\right)\left\langle f\,(\cdot,z)\,b\,(\cdot)\,\frac{\partial\phi\,(\cdot,z)}{\partial y}\right\rangle\left(A^1\,(\tau,k,l,z)\right)^2\\
&\quad+\left(\rho_{su}\rho_{vy}+\rho_{vu}\rho_{sy}\right)\sigma_s\sigma_v\,(kl)\left\langle b\,(\cdot)\,\frac{\partial\psi\,(\cdot,z))}{\partial y}\right\rangle A^1\,(\tau,k,l,z)\,,
\end{aligned}
$$

5.3. The functions $B^{n\,|\,01},\ n=0,1,2$, $R^{1\,|\,01}(\tau,k,l,z)$ and $R^{2\,|\,01}(\tau,k,l,z)$ of the first order term P_{01} of the series expansion (4.1)

$$B^{2\,|\,01}(\tau,k,l,z)=\int_0^\tau e^{2(\tau-s)\zeta(k,l,z)}\left(\frac{1-\xi(k,l,z)e^{s\,\zeta(k,l,z)}}{1-\xi(k,l,z)e^{\tau\,\zeta(k,l,z)}}\right)^4 R^{2\,|\,01}(s,k,l,z)\,ds,$$

$$B^{1\,|\,01}(\tau,k,l,z)=\int_0^\tau e^{(\tau-s)\zeta(k,l,z)}\left(\frac{1-\xi(k,l,z)e^{s\,\zeta(k,l,z)}}{1-\xi(k,l,z)e^{\tau\,\zeta(k,l,z)}}\right)^2 R^{1\,|\,01}(s,k,l,z)\,ds,$$

$$B^{0\,|\,01}(\tau,k,l,z)=\kappa m\int_0^\tau B^{1\,|\,01}(s,k,l,z)ds$$

$$\begin{aligned}
R^{2\,|\,01}(\tau,k,l,z)&=\Big(\tilde{V}_{uz}A^1(\tau,k,l,z)-\tilde{V}_{sz}(ik)-\tilde{V}_{vz}(il)\Big)\frac{\partial A^1}{\partial\tilde{\sigma}}(\tau,k,l,z)\\
&\quad+\big(\bar{V}_{uz}A^1(\tau,k,l,z)-\bar{V}_{sz}(ik)-\bar{V}_{vz}(il)\big)\frac{\partial A^1}{\partial\bar{\sigma}}(\tau,k,l,z)\\
R^{1\,|\,01}(\tau,k,l,z)&=\Big(\tilde{V}_{uz}A^1(\tau,k,l,z)-\tilde{V}_{sz}(ik)-\tilde{V}_{vz}(il)\Big)\frac{\partial A^0}{\partial\tilde{\sigma}}(\tau,k,l,z)\\
&\quad+\big(\bar{V}_{uz}A^1(\tau,k,l,z)-\bar{V}_{sz}(ik)-\bar{V}_{vz}(il)\big)\frac{\partial A^0}{\partial\bar{\sigma}}(\tau,k,l,z)\\
&\quad+\left(\tilde{V}_{uz}\frac{\partial A^1}{\partial\tilde{\sigma}}(\tau,k,l,z)+\bar{V}_{uz}\frac{\partial A^1}{\partial\bar{\sigma}}(\tau,k,l,z)\right)\\
&\quad+\big(2\kappa m+\bar{\sigma}^2(z)\big)\,B^{2\,|\,01}(\tau,k,l,z),
\end{aligned}$$

$$\tilde{V}_{sz}(z)=\rho_{sz}\sigma_s d(z)\tilde{\sigma}'(z),\quad \tilde{V}_{vz}(z)=\rho_{vz}\sigma_v d(z)\tilde{\sigma}'(z),\quad \tilde{V}_{uz}(z)=\rho_{uz}d(z)\tilde{\sigma}(z)\tilde{\sigma}'(z)$$
$$\bar{V}_{sz}(z)=\rho_{sz}\sigma_s d(z)\bar{\sigma}'(z),\quad \bar{V}_{vz}(z)=\rho_{vz}\sigma_v d(z)\bar{\sigma}'(z),\quad \bar{V}_{uz}(z)=\rho_{uz}d(z)\tilde{\sigma}(z)\bar{\sigma}'(z)$$

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